Tut-1

Electromagnetic waves (5 marks)

* Maxwell's eqn (Integral form)

→ Maxwell discovered that the basic principle of electromagnetism can be expressed in terms of 4 equation which are described below:

1> Gauss law of electrostatics:

It states that, "the total flux through closed surface enclosing a charge 'q' is equal to $1/\varepsilon_0$ times magnitude of charge enclosed."

i.e. $\oint \vec{E}\cdot d\vec{A} = q/\varepsilon_0$

This relates the electric field & charge distribution. It confirms the existence of single charge i.e. monopole exist.

2> Gauss law for magnetism:

⇒ It states that, "the total mag. flux through closed surface is zero."

i.e. $\oint \vec{B}\cdot d\vec{A} = 0$

It confirms that magnetic monopole does not exist

3> Faraday's law of E.M.I:

→ It states that, "induced emf in circuit is equal to the rate of change of magnetic field i.e. $\oint \vec{E} \cdot \vec{dl} = -\frac{d\phi_B}{dt}$

It says that changing magnetic field with time produces an electric field.

4> Amphere Maxwell's law:

→ It is modification of amphere's law by Maxwell. It describes that there are two ways of setting a magnetic field

i> By means of steady current (Amphere's law):

i.e. $\oint \vec{B} \cdot \vec{dl} = \mu_0 I$

ii> By means of changing electric field (Maxwell law of Induction)

i.e. $\oint \vec{B} \cdot \vec{dl} = \mu_0 \varepsilon_0 \frac{d\phi_E}{dt}$

The combined form of these two eqns is Amphere Maxwell law.

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\phi_E}{dt}$$

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 \left(I + \varepsilon_0 \frac{d\phi_E}{dt} \right)$$

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 (I + I_d)$$

is displacement current.

* Maxwell's eq^n in differential form:

i> Maxwell's first equation:

From Gauss, law of electrostatics, we have,

$$\oint \vec{E} \cdot d\vec{A} = q/\varepsilon_0$$

$$\oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} \oint \rho \, dV$$

Now, using gauss-divergence theorem,

$$\oint (\vec{\nabla} \cdot \vec{E}) \cdot d\vec{v} = \frac{1}{\varepsilon_0} \oint \rho \, dV$$

$$\text{or, } \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{— (I)}$$

ii> Maxwell's second equation:

From Gauss law of magnetism, we have,

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Now using, Gauss-divergence theorem,

$$\oint (\vec{\nabla} \cdot \vec{B}) \cdot d\vec{V} = 0$$

$$\boxed{\nabla \cdot \vec{B} = 0} \quad \text{— (II)}$$

iii> Maxwell's 3rd equation,

According to Faraday's law of E.M.I, we have

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\phi_B}{dt}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oint B \cdot d\vec{A}$$

Now using curl-stokes theorem,

$$\cancel{\nabla \times d\vec{A}} \Rightarrow \oint (\nabla \times \vec{E}) \cdot d\vec{A} = -\frac{d}{dt} \oint B \cdot d\vec{A}$$

$$\boxed{\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}} \text{ --- (III)}$$

iv> Maxwell's 4th equation,

According to Amphere's Maxwell law, we have,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 (I + I_d)$$

where, $I_d = \varepsilon_0 \frac{d\phi_E}{dt}$

& real current, $I = \oint \vec{J} \cdot d\vec{A}$

or, $\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(\oint \vec{J} \cdot d\vec{A} + \varepsilon_0 \frac{d}{dt} \oint \vec{E} \cdot d\vec{A} \right)$

or, $\oint \vec{B} \cdot d\vec{l} = \mu_0 \left[\oint \left(\vec{J} + \varepsilon_0 \frac{d\vec{E}}{dt} \right) \right] d\vec{A}$

By using curl-stokes theorem.

$$\oint (\nabla \times \vec{B}) \cdot d\vec{A} = \mu_0 \left[\oint d\vec{A} \left(\vec{J} + \varepsilon_0 \frac{d\vec{E}}{dt} \right] \right.$$

$$\boxed{\therefore \nabla \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{d\vec{E}}{dt} \right)} \text{ --- (IV)}$$

Electromagnetic wave eqn in free space:

In free space, the charge density 'ρ' & current density J is zero.

∴ Maxwell's eqn becomes,

$$\nabla \cdot \vec{E} = 0 \quad \text{— (I)}$$

$$\nabla \cdot \vec{B} = 0 \quad \text{— (II)}$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \quad \text{— (III)}$$

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{d\vec{E}}{dt} \quad \text{— (IV)}$$

Taking curl on both sides, in eqn (III)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{d}{dt}(\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{E}(\vec{\nabla} \cdot \vec{\nabla}) = -\frac{d}{dt}\left(\mu_0 \varepsilon_0 \frac{d\vec{E}}{dt}\right)$$

$$-\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{d^2 \vec{E}}{dt^2}$$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{d^2 E}{dt^2}} \quad \text{— (V)}$$

Taking curl of eqn (IV), & proceeding in similar way, we get,

$$\boxed{\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{d^2 \vec{B}}{dt^2}} \quad \text{— (VI)}$$

Eqn (V) & (VI) are equation of electromagnetic wave, now compare this eqn with general wave eqn.

$$\nabla^2 y = \frac{1}{v^2}\frac{d^2y}{dt^2}$$

$$\therefore v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$v = 3\times10^8$ m/s

This is same as velocity of light in vacuum.

Electromagnetic eqn in nonconducting / dielectric medium

~~In free space,~~ In dielectric medium, the charge 'ρ' & current density (J) is zero.

If medium have permittivity μ then Maxwell's eqn becomes:

$$\nabla\cdot\vec{E} = 0 \quad -(i)$$

$$\nabla\cdot\vec{B} = 0 \quad -(ii)$$

$$\nabla\times\vec{E} = -\frac{d\vec{B}}{dt} \quad -(iii)$$

$$\nabla\times\vec{B} = \mu\varepsilon_0\frac{d\vec{E}}{dt} \quad -(iv)$$

Taking curl on both sides in eqn (iii),

$$\nabla\times\nabla\times\vec{E} = -\frac{d}{dt}(\vec{\nabla}\times\vec{B})$$

$$\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{E}) - \vec{E}(\vec{\nabla}\cdot\vec{\nabla}) = -\frac{d}{dt}\left(\mu\varepsilon\frac{d\vec{E}}{dt}\right)$$

$$-\nabla^2\vec{E} = -\mu\varepsilon\frac{d^2\vec{E}}{dt^2}$$

$$\boxed{\nabla^2\vec{E} = \mu\varepsilon\frac{d^2E}{dt^2}} \quad -(v)$$

Taking curl of eqⁿ (IV), & proceeding in similar way, we get,

$$\boxed{\nabla^2 \vec{B} = \mu \varepsilon \frac{d^2 \vec{B}}{dt^2}} \quad - (VII)$$

eqⁿ (V) & (VI) are eqⁿ of electromagnetic wave, comparing this eqⁿ with a general we get,

$$\Delta^2 y = \frac{1}{v^2} \frac{d^2 y}{dt^2}$$

$$\boxed{\therefore v = \frac{1}{\sqrt{\mu \varepsilon}}}$$

which is req relation

E.M wave eqn in conducting (isotropic medium):

→ For conduction medium, the charge density 'ρ' & current density $\vec{J}$ wont be zero. Then Maxwell's eqⁿ becomes,

$$\nabla \cdot \vec{E} = \rho / \varepsilon_0 \quad - (I)$$

$$\nabla \cdot \vec{B} = 0 \quad - (II)$$

$$\nabla \times \vec{E} - \frac{d\vec{B}}{dt} \quad - (III)$$

$$\nabla \times \vec{B} = \mu_0 \left| \vec{J} + \varepsilon \frac{d\vec{E}}{dt} \right| - (IV)$$

Taking curl on eqⁿ (III),

$$\nabla \times \nabla \times \vec{E} = -\frac{d}{dt}(\nabla \times \vec{B})$$

$$\nabla(\nabla \cdot \vec{E}) - \vec{E}(\nabla \cdot \nabla) = \cancel{-\frac{d}{dt}\frac{d\vec{E}}{dt}} \; -\frac{d}{dt}\left[\mu\left(\vec{J} + \varepsilon\frac{d\vec{E}}{dt}\right)\right]$$

For medium of constant charge density

$$\nabla(\nabla \cdot \vec{E}) = 0$$

$$\therefore \; 0 - \nabla^2 \vec{E} = -\frac{d}{dt}\left[\mu\left(\vec{J} + \varepsilon\frac{d\vec{E}}{dt}\right)\right]$$

$$\text{or, } \nabla^2 \vec{E} = \frac{d}{dt}\left[\mu\left(\sigma\vec{E} + \varepsilon\frac{d\vec{E}}{dt}\right)\right] \qquad \vec{J} = \sigma\vec{E}$$

$$\therefore \; \nabla^2 \vec{E} = \mu\sigma\frac{d\vec{E}}{dt} + \mu\varepsilon\frac{d^2\vec{E}}{dt^2} \quad \text{— (V)}$$

Similarly, taking curl on eqⁿ (VI).

$$\therefore \; \nabla^2 \vec{B} = \mu\sigma\frac{d\vec{B}}{dt} + \mu\varepsilon\frac{d^2\vec{B}}{dt^2} \quad \text{— (VI)}$$

eqⁿ (V) & (VI) are E.M ~~equations~~ waves in conducting medium.

Displacement current (I_d):

The ampere Maxwell law is

$$\oint \vec{B}\cdot\vec{dl} = \mu_0\left(I + \cancel{\mu_0}\,\varepsilon_0 \frac{d\phi_E}{dt}\right) \quad \text{-(I)}$$

If we compare two terms on right side of this eqn, it is see that the product $\varepsilon_0 \frac{d\phi_E}{dt}$ must have the dimension of current. This product is considerd as fictitious current associated with the changing electric field betn the plate of capacitor is called displacement current.

we can now rewrite ampere Maxwell law as,

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 (I + I_d) \quad \text{-(II)}$$

$$\text{where, } I_d = \varepsilon_0 \frac{d\phi_E}{dt}$$

The charge stored in parallel plate capacitor,

$$q = CV$$

$$= \frac{\varepsilon_0 A}{d} V$$

$$\therefore q = \varepsilon_0 A E$$

$$\text{real current } (I) = \frac{dq}{dt} = \varepsilon_0 A \frac{dE}{dt} \quad \text{-(III)}$$

Here,

$$\text{displacement current } (I_d) = \varepsilon_0 \frac{d\phi_E}{dt} = \varepsilon_0 \frac{d}{dt}(EA)$$

$$\therefore I_d = \varepsilon_0 A \frac{dE}{dt} \quad \text{-(IV)}$$

From eq^n (iii) & (iv), it is seen that the real current (I) during charging and discharging of capacitor is equal to displacement current 'I_d' betn plates.

3) Q. Calculate the displacement current betn square plates of capacitor having 1 side 1cm. if electric field betn plates is changing at the rate of 3×10^6 V/m.sec.

Soln

$I_d = ?$

$\frac{dE}{dt} = 3\times10^6$

$I_d = 8.85\times10^{-12}\times(1\times10^{-12})^2\times3\times10^6$ Ans, $\left(I_d = \epsilon_0 A \frac{dE}{dt}\right)$

$= 2.65\times10^{-9}$ A

* Energy transport & poynting vector: IMP

⇒ The magnitude of poynting vector is defined as rate of energy transport per unit area in plane electromagnetic wave.

$$\boxed{\text{ie. } S = \frac{1}{A}\frac{dU}{dt}}$$

Consider a propagation of electromagnetic wave in a box of area 'A' & thickness 'du' at any instant. The energy stored in a box is given by,

$$dU = dU_E + dU_B \qquad \mu = \frac{E}{V}$$

$$dU = \mu_E A\,du + \mu_B A\,du \qquad \mu_E = \frac{dU_E}{A\,du}$$

where,

$\mu_E = \frac{1}{2}\varepsilon_0 E^2$ is energy density in electric field.

$\mu_B = \frac{B^2}{2\mu_0}$ is energy density in magnetic field.

$$\therefore dU = \left|\frac{1}{2}\varepsilon_0 E^2 + \frac{B^2}{2\mu_0}\right| A\,du \qquad \left[\frac{E}{B} = c\right]$$

$$\text{or, } dU = \left|\frac{1}{2}\varepsilon_0 E^2 + \frac{E^2}{2\mu_0 c^2}\right| A\,du$$

$$\text{or, } dU = \left|\frac{1}{2}\varepsilon_0 E^2 + \frac{E^2}{2\mu_0}\mu_0\varepsilon_0\right| A\,du$$

$$\text{or, } dU = \varepsilon_0 E^2 A\,du$$

$$\therefore \boxed{S = \varepsilon_0 E^2 c}$$

$$\text{or, } S = \varepsilon_0 E B c^2$$

$$\boxed{\therefore S = \frac{EB}{\mu_0}}$$

Above relation can be written as,

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

The intensity of E.M wave can be defined as average of poynting vector.

ie. $$\boxed{I = \frac{E_0 B_0}{2\mu_0}}$$

* Radiation Pressure: (P_r):

⇒ E.M waves have linear momentum as well as energy. This means E.M wave can exert a pressure when incident on an object. The force per unit area on an object due to E.M radiation is called radiation pressure.

Let a beam of E.M radiation of intensity 'I' is incident on an object of area 'A' perpr to path of ~~direction~~ radiation. The object is free to move & energy (ΔU) of radiation is entirely absorbed by the object in time interval ΔT. The energy absorbed by area 'A' is, ~~(ΔU)~~

$$\Delta U = IA\,dt$$

The change in momentum 'ΔP' of object related to change in energy (ΔU) is given by,

$$\Delta P = \frac{\Delta U}{c}$$

$$\Delta P = \frac{IA\,dt}{c}$$

Also, from Newton's 2nd law of motion,
change in momentum is related to the force as,

$$F = \frac{\Delta P}{\Delta t} = \frac{IA}{c}$$

$$\therefore \frac{F}{A} = \frac{I}{c}$$

$$\boxed{\therefore \text{Radiation Pressure } (P_r) = \frac{I}{c}}$$

Instead of being absorbed, if radiation is entirely reflected back along its original path, the change in momentum is given by, $\Delta P = 2\frac{\Delta U}{c}$

Proceeding in similar way,

The radiation pressure in this can be calculated as,

$$\boxed{P_r = \frac{2I}{c}}$$

* Relation betn E_0 & B_0 ie. $c = \frac{E_0}{B_0}$

we have, 3rd Maxwell eqn as,

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

In 1D, it can be expressed as,

$$\frac{d\vec{E}}{dx} = -\frac{d\vec{B}}{dt} \quad - (1)$$

we have,

$$E = E_0 \sin(kx - \omega t)$$

Now, diff this eq^n w.r.t x,

$$\frac{dE}{dx} = E_0 k \cancel{\sin} \cos(kx - \omega t) \quad \text{--- (ii)}$$

& $B = B_0 \sin(kx - \omega t)$

differentiating this eq^n w.r.t 't',

$$\frac{dB}{dt} = -B_0 \omega \cos(kx - \omega t) \quad \text{--- (iii)}$$

Now, from (i), (ii) & (iii),

$$E_0 k \cancel{\cos(kx-\omega t)} = -B_0 \omega \cancel{\cos(kx-\omega t)}$$

or, $$\frac{E_0}{B_0} = -\frac{\omega}{k}$$

$$\boxed{\therefore \frac{E_0}{B_0} = c}$$

* Charge conservation theorem (continuity eq^n)

~~or~~ $$\boxed{\nabla \cdot \vec{J} + \frac{d\rho}{dt} = 0}$$

we have, 4th Maxwell eq^n,

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{d\vec{E}}{dt} \right)$$

Taking divergence on both sides,

$$\nabla \cdot \left\{ (\nabla \times \vec{B}) \right\} = \mu_0 \left(\nabla \cdot \vec{J} + \frac{\epsilon_0 d}{dt} (\nabla \cdot \vec{E}) \right)$$

~~or,~~ Since divergence of curl of any vector is zero.

ie. $\nabla \cdot (\nabla \times \vec{B}) = 0$.

Now,
$$\bar{\nabla}\cdot\vec{J} + \varepsilon_0 \frac{d}{dt}(\bar{\nabla}\cdot\vec{E})$$

$$\bar{\nabla}\cdot J + \varepsilon_0 \frac{d(\rho/\varepsilon_0)}{dt}$$

$$\boxed{\therefore\ \bar{\nabla}\cdot\vec{J} + \frac{d\rho}{dt}} = 0.$$